ACTIVITY CODE: 1903121021

B.Sc. 6th Semester (Honours) Examination, October 2020

Subject: Mathematics

Course ID: 62112 **Course Code: SH/MTH/602/C-14**

Course Title: Ring Theory and Linear Algebra II

Full Marks: 20 **Time: 1 Hour**

<u>The figures in the margin indicate full marks</u>

Notations and symbols have their usual meanings.

1. **Answer *any two* from the following questions:** **2×2=4**

a) Give an example of an Integral Domain which is not a Principal ideal domain (PID).

b) State Eisenstein's Criterion for irreducibility of a polynomial in $\mathbb{Z}[x]$.

c) In a commutative ring with 1, give an example of a prime element which is not irreducible.

d) Define the dual space of a vector space V over a field F.

e) Give an example of a linear operator whose characteristic polynomial and minimal polynomial are not equal.

f) Show that the minimal polynomial of a square matrix is unique.

g) Establish the law of parallelogram in an inner product space.

h) In an inner product space, prove that an orthogonal set of non-zero vectors is linearly independent.

2. **Answer *any two* from the following questions:** **5×2=10**

a) In an integral domain, prove that any prime element is irreducible. Show that the converse of this result is not true. **2+3=5**

b) State division algorithm for polynomial rings over a commutative ring with 1. Find the gcd of the polynomials $f(x) = x^4 + 3x^3 - 3x + 1$, $g(x) = x^3 + 3x^2 + x - 2$ in $\mathbb{Q}[x]$. **1+4=5**

c) Let V be a finite dimensional vector space over the field F. Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis for V. Show that there exists a unique dual basis B^* for the dual space V^* corresponding to B. **5**

d) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. **5**

e) Consider the vector space $\mathbb{R}^3$ over $\mathbb{R}$ equipped with the standard inner product. Applying Gram-Schmidt orthogonalization process, orthonormalize the following set of vectors: $\{(1,2,-2),(2,0,1),(1,1,0)\}$. **5**

f) Let V be a finite dimensional inner product space and f be a linear functional on V. Then prove that there exists a unique vector β in V such that $f(\alpha) = <\alpha, \beta>$ for all α ∈ V. **5**

3. Answer *any two* from either a) or b): **3×2=6**

a) (i) Show that $\mathbb{Z}[x]$ is not a Principal ideal domain (PID) by exhibiting an ideal of $\mathbb{Z}[x]$ which is not a principal ideal.

(ii) Let T be a linear operator on a vector space V over F such that $T^2 = T$. Prove that T is diagonalizable.

(iii) Prove that Inner product function of an Inner product space is a continuous function.

(iv) Let V be a complex or real inner product space. Then show that the induced norm satisfies the following equality:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

for all x, y in V.

b) (i) Show that the polynomial $f(x) = 2x^4 + 6x^3 - 9x^2 + 15$ is irreducible over $\mathbb{Z}$.

(ii) Find all possible non-similar 4×4 real matrices with minimal polynomial x^2.

(iii) Let T be a linear operator on a complex inner product space V. Define T^* (***adjoint of T***). For any scalar c, prove that $(c\,T)^* = \bar{c}T^*$.

(iv) Prove that every Euclidean domain is a principal ideal domain.

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